

PROJECTIVE MODULES AND YOUNG'S SEMINORMAL FORM.

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ABSTRACT. We study the representation theory of the symmetric group S_n in positive characteristic p . Using features of the LLT-algorithm we give for $n < p^2$ a conjectural description of the projective cover $P(\lambda)$ of the simple module $D(\lambda)$ where λ is a p -restricted partition. Inspired by the recent theory of Khovanov-Lauda-Rouquier algebras we explain an algorithm that allows us to verify the conjecture for $n \leq 15$, at least.

1. INTRODUCTION.

In this paper we continue the investigation from [RH1-3] that seeks to demonstrate the relevance of Young's seminormal form for the representation theory of the symmetric group S_n in characteristic $p > 0$.

Let p be a prime and let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ be the finite field of p elements. Let $\text{Par}_{res}(n)$ denote the set of p -restricted partitions of n . In [RH3] we proved that the simple $\mathbb{F}_p S_n$ -module $D(\lambda)$ given by $\lambda \in \text{Par}_{res}(n)$, is generated by $\overline{a_\lambda E_\lambda}$ where $E_\lambda \in \mathbb{Q}S_n$ is the Jucys-Murphy idempotent associated with λ and $a_\lambda \in R$ is the least common multiple of the denominators of the coefficients of E_λ when it is expanded in the natural basis of $\mathbb{Q}S_n$. Note that although $a_\lambda E_\lambda$ is a preidempotent, its reduction modulo p , that is $\overline{a_\lambda E_\lambda}$, will in general have zero square and will therefore not be a preidempotent. This corresponds to the fact that in general $D(\lambda)$ is not a projective module for $\mathbb{F}_p S_n$.

Hence we are lead to ask if it might be possible to describe the projective cover $P(\lambda)$ of $D(\lambda)$ within the theory of Young's seminormal form, and indeed the present paper is dedicated to this question.

Let us briefly explain our results. For all $\lambda \in \text{Par}_{res}(n)$, such that all ladders of the corresponding ladder tableau are of length less than p , we construct an idempotent $\tilde{e}_\lambda \in \mathbb{F}_p S_n$. The main ingredients for this construction are Murphy's tableau class idempotents for the ladder class of λ , and a symmetrization procedure over the ladder group. We observe that under the condition $n < p^2$ of James' Conjecture for the adjustment matrix of the decomposition matrix, all ladder lengths as above are automatically less than p .

Defining $\widetilde{A(\lambda)} := \mathbb{F}_p S_n \tilde{e}_\lambda$ we do obtain a projective $\mathbb{F}_p S_n$ -module, but it is decomposable in general, that is $A(\lambda) \neq P(\lambda)$. On the other hand, we show that there is triangular expansion of the form,

$$\widetilde{A(\lambda)} = P(\lambda) \oplus \bigoplus_{\mu, \mu \triangleright \lambda} P(\mu)^{\oplus m_{\lambda\mu}} \quad (1)$$

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where \triangleleft is the usual dominance order on partitions.

Recall the Lascoux-Leclerc-Thibon (LLT) algorithm that gives a way of calculating the global crystal basis $\{G(\lambda) \mid \lambda \in \text{Par}_{\text{res}}(n)\}$, of the basic submodule \mathcal{M}_q of the q -Fock space \mathcal{F}_q . An important tool for this algorithm is given by certain combinatorially defined elements $A(\lambda) \in \mathcal{M}_q$ called “the first approximation of the global basis” in [LLT]. They have the following triangular expansion property

$$A(\lambda) = G(\lambda) + \sum_{\mu, \mu \triangleright \lambda} n_{\lambda\mu} G(\mu)$$

where $n_{\lambda\mu} \in \mathbb{Z}[q, q^{-1}]$, and $\overline{n_{\lambda\mu}} = n_{\lambda\mu}$, where $\bar{\cdot}$ is given by $q \mapsto q^{-1}$. The LLT-algorithm is a recursive procedure based on this.

Our main point is now to consider $A(\lambda)$ as an object of interest in itself, and not just a tool for calculating $G(\lambda)$. In this spirit, for $n < p^2$ we conjecture that $A(\lambda)$ should be categorified by $\widetilde{A(\lambda)}$, or to be more precise that

$$n_{\lambda\mu}(1) = m_{\lambda\mu}. \quad (2)$$

An important motivation for studying this Conjecture is contained in our Theorem 4 below, showing that if it is true, then James’ Conjecture follows by inverting equation (1).

In section 5 of the paper we describe a method for verifying formula (2), which we believe is of independent interest. It is based on the isomorphism between $\mathbb{F}_p S_n$ and \mathcal{R}_n , the cyclotomic Khovanov-Lauda-Rouquier (KLR) algebra of type A , that was proved by Brundan and Kleshchev in [BK]. In this situation, \tilde{e}_λ is closely related to the KLR-idempotents and so, with $S(\mu)$ denoting the Specht module, we get that $\text{Hom}_{\mathbb{F}_p S_n}(\widetilde{A(\lambda)}, S(\mu)) = \tilde{e}_\lambda S(\mu)$ identifies with the symmetrized generalized eigenspace for the action of the Jucys-Murphy elements in $S(\mu)$. Let $\langle \cdot, \cdot \rangle_\mu$ be the symmetric, S_n -invariant, bilinear form on $S(\mu)$, given by the cellular algebra structure on $\mathbb{F}_p S_n$. Then for $\lambda \in \text{Par}_{\text{res}}(n)$ we have $D(\lambda) = S(\lambda)/\text{rad}\langle \cdot, \cdot \rangle_\lambda$, and so $\dim D(\lambda) = \text{rank}\langle \cdot, \cdot \rangle_\lambda$, where $\text{rank}\langle \cdot, \cdot \rangle_\lambda$ is the rank of the matrix associated with the form. Unfortunately, $\dim S(\mu)$ grows fast with respect to n . Already for $n \approx 20$ we have $\dim S(\mu)$ at magnitudes of several millions, and even fast computers will in general not be able to calculate the rank. On the other hand, the eigenspaces $\tilde{e}_\lambda S(\mu)$ have much smaller dimensions, for example less than 10 for $n \approx 20$ and $p = 5$.

Still, this is not quite enough to perform calculations. As a matter of fact, in order to do them we need to rely on our recent results from [RH3] on the compatibility of the “intertwining elements” from Brundan and Kleshchev’s work with Young’s seminormal form. They allow us to describe the action of the KLR-generators ψ_i completely in terms of Young’s seminormal form, and thus to calculate the rank of $\langle \cdot, \cdot \rangle_\lambda$ on $\tilde{e}_\lambda S(\mu)$. For $n \approx 20$ and $p = 5$, our GAP-implementation calculates the individual ranks in less than one second. Our partial verification of the conjectural formula (2) follows from these calculations.

2. BASIC NOTATION

Let p be a prime and let R be the localization of \mathbb{Z} at p . Let S_n be the symmetric group on n letters and write $\sigma_i := (i-1, i)$. We are interested in the representation theory of S_n over the finite field $\mathbb{F}_p = R/pR$.

Over \mathbb{Q} , the irreducible representations of S_n are parametrized by the set $\text{Par}(n)$ of partitions of n , that is the set of nonincreasing sequences of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with sum n . Over \mathbb{F}_p they are parametrized by the set of p -restricted partitions $\text{Par}_{\text{res}}(n)$, consisting of those $\lambda \in \text{Par}(n)$ that satisfy $\lambda_i - \lambda_{i+1} < p$ for all i where by convention $\lambda_i = 0$ for $i \geq k+1$. For $\lambda \in \text{Par}(n)$ we denote by $S(\lambda)$ the Specht module for S_n , defined via Murphy's standard basis, see [Mu95]. In general, if M is an RS_n -module, we can reduce it modulo p to obtain $\overline{M} := M \otimes_R \mathbb{F}_p$, an $\mathbb{F}_p S_n$ -module. Then the standard basis is also a basis for $\overline{S(\lambda)}$. Since it is a cellular basis in the sense of Graham and Lehrer [GL], $\overline{S(\lambda)}$ is equipped with a bilinear, symmetric S_n -invariant form $\langle \cdot, \cdot \rangle_\lambda$, which is nonzero iff $\lambda \in \text{Par}_{\text{res}}(n)$. We obtain the parametrization of the simple modules for $\mathbb{F}_p S_n$ via $\lambda \in \text{Par}_{\text{res}}(n) \mapsto D(\lambda) := \overline{S(\lambda)} / \text{rad} \langle \cdot, \cdot \rangle_\lambda$.

In the paper we shall be specially interested in the projective covers of the simple modules. For $\lambda \in \text{Par}_{\text{res}}(n)$ we let $P(\lambda)$ denote the projective cover of $D(\lambda)$. By definition, $P(\lambda)$ is the unique indecomposable projective $\mathbb{F}_p S_n$ -module such that $D(\lambda)$ is a homomorphic image of $P(\lambda)$. By general theory, $P(\lambda)$ is of the form $P(\lambda) = \mathbb{F}_p S_n e_\lambda$ for some idempotent $e_\lambda \in \mathbb{F}_p S_n$.

Recall that a partition $\lambda = (\lambda_1, \dots, \lambda_k) \in \text{Par}(n)$ is represented graphically via its "Young diagram". It consists of k , left aligned, files of boxes, called nodes, in the plane, with the first file containing λ_1 nodes, the second file containing λ_2 nodes and so on. The nodes are indexed using matrix convention, with the $[i, j]$ 'th node situated in the j 'th column of the i 'th file. For $\lambda \in \text{Par}(n)$, a λ -tableau t is a filling of the nodes of λ with the numbers $\{1, 2, \dots, n\}$. We write $t[i, j] = k$ if the $[i, j]$ 'th node of t is filled with k and $c_t(k) = j - i$ if $t[i, j] = k$. Then $c_t(k)$ is called the content of t at k , whereas its image in \mathbb{F}_p , denoted $r_t(k)$, is called the residue of t at k . For $k \in \{1, 2, \dots, n\}$ we define $t(k) := [i, j]$ where $t[i, j] = k$. A tableau t is called standard if $t[i, j] \leq t[i, j+1]$ and $t[i, j] \leq t[i+1, j]$ for all relevant i, j . The set of standard tableaux of partitions of n is denoted $\text{Std}(n)$ and the set of standard tableaux with underlying partition λ is denoted $\text{Std}(\lambda)$. For $\lambda \in \text{Par}(n)$ and $t \in \text{Std}(\lambda)$ -tableau we write $\text{Shape}(t) := \lambda$.

Let t be a λ -tableau with node $[i, j]$. The $[i, j]$ -hook consists of the nodes of the Young diagram of λ situated to the right and below the $[i, j]$ node and its cardinality is called the hook-length h_{ij} . The product of all hook-lengths is denoted h_λ . The hook-quotient of the tableau $t \in \text{Std}(\lambda)$ at n is the number $\gamma_{tn} = \prod \frac{h_{ij}}{h_{ij}-1}$ where the product is taken over all nodes in the row of λ that contains n , omitting hooks of length one. For a general i , we define γ_{ti} similarly, by first deleting from t the nodes containing $i+1, i+2, \dots, n$. Finally we define $\gamma_t = \prod_{i=2}^n \gamma_{ti}$.

Let us recall the combinatorial concepts of ladders and ladder tableaux that play an important role for the LLT-algorithm, although we shall use

conventions that are dual to the ones of [LLT]. Let μ be a p -restricted partition. The 'ladders' of μ are the straight 'line segments' through the Young diagram of μ with 'slope' $1/(p-1)$, that is the subsets of the nodes of μ of the form $\mathcal{L}_b := \{[i, j] \mid j = b - (p-1)(i-1)\}$. If $\mu \in \text{Par}_{res}(n)$ we have that the ladders are 'unbroken', that is $\pi_1(\mathcal{L}_a)$ is of the form $\{q, q+1, a+2, \dots, r\}$ for some $q < r$ where π_1 is the first projection. We say that \mathcal{L}_b is smaller than \mathcal{L}_{b_1} if $b < b_1$. The ladder tableau μ_{lad} of μ is defined as the μ -tableau with the numbers $1, 2, \dots, n$ filled in one ladder at the time, starting with the smallest ladder and continuing successively upwards, the numbers being filled in from top to bottom in each ladder. Note that the residues are constant on each ladder.

For a partition μ , the p -residue diagram res_μ is obtained by writing the residue $r_t(k)$ in the $[i, j]$ 'th node of the Young diagram of μ . For example, if $\mu = (6, 5, 3, 1)$ and $p = 3$ then the residue diagram and ladder tableau are as follows

$$res_\mu = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 & 2 \\ \hline 2 & 0 & 1 & 2 & 0 & \\ \hline 1 & 2 & 0 & & & \\ \hline 0 & & & & & \\ \hline \end{array}, \quad \mu_{lad} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 7 & 10 \\ \hline 4 & 6 & 8 & 11 & 13 & \\ \hline 9 & 12 & 14 & & & \\ \hline 15 & & & & & \\ \hline \end{array}$$

with ladders $\mathcal{L}_1 = \{1\}$, $\mathcal{L}_2 = \{2\}$, $\mathcal{L}_3 = \{3, 4\}$, $\mathcal{L}_4 = \{5, 6\}$, $\mathcal{L}_5 = \{7, 8, 9\}$, $\mathcal{L}_6 = \{10, 11, 12\}$ and $\mathcal{L}_7 = \{13, 14, 15\}$. We denote by $\underline{i}_{lad, \mu}$, the residue sequence given by the ladder tableau for μ . In the above example it is

$$\underline{i}_{lad, \mu} = (0, 1, 2, 2, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 0).$$

The ladders define a sequence of subpartitions $\mu_{lad, \leq 1}, \dots, \mu_{lad, \leq m}$ of μ where $\mu_{lad, \leq k}$ is defined as the union of the ladders $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_k$.

The dominance order \trianglelefteq on partitions is defined by

$$\lambda \trianglelefteq \mu \text{ if } \sum_{i=1}^m \lambda_i \leq \sum_{i=1}^m \mu_i \text{ for } m = 1, 2, \dots, \min(k, l)$$

for $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$. When λ is used as a subscript where a tableau is expected, it is referring to the unique maximal λ -tableau t^λ , having the numbers $\{1, \dots, n\}$ filled in along the rows. The dominance order extends to tableaux by considering them as series of partitions. Following Murphy in [Mu83], we define an equivalence relation on the set of all standard tableaux via $t \sim_p s$ if $r_t(k) = r_s(k) \bmod p$ for all k . The classes of \sim_p are called tableaux classes, they are parametrized by \mathbb{F}_p^n . The ladder tableaux are 'minimal' in their classes in the sense of the following Lemma.

Lemma 1. *Assume that λ is p -restricted. Then if $t \in [\lambda_{lad}]$ we have that either $\text{Shape}(t) \triangleright \lambda$ or $\text{Shape}(t) = \lambda$ and $t \trianglelefteq \lambda_{lad}$.*

Proof. Omitted. □

For $k = 1, 2, \dots, n$ the Jucys-Murphy elements $L_k \in \mathbb{Z}S_n$ are defined by

$$L_k := (1, k) + (2, k) + \dots + (k-1, k)$$

with the convention that $L_1 := 0$. An important application of the L_k is the construction of orthogonal idempotents $E_t \in \mathbb{Q}S_n$, indexed by tableaux

t , from which Young's seminormal form can be derived. We denote these idempotents the Jucys-Murphy idempotents. Their construction is as follows

$$E_t := \prod_{\{c \mid -n < c < n\}} \prod_{\{i \mid c_t(i) \neq c\}} \frac{L_i - c}{c_t(i) - c}.$$

For t standard we have $E_t \neq 0$, whereas for t nonstandard either $E_t = 0$, or $E_t = E_s$ for some standard tableau s related to t . Running over all standard tableaux, the E_t form a set of primitive and complete idempotents for $\mathbb{Q}S_n$, that is their sum is 1. Moreover, they are eigenvectors for the action of the Jucys-Murphy operators in $\mathbb{Q}S_n$, since

$$(L_k - c_t(k))E_t = 0 \text{ or equivalently } L_k = \sum_{t \in \text{Std}(n)} c_t(k)E_t. \quad (3)$$

For $\lambda \in \text{Par}(n)$, we let Stab_λ denote the row stabilizer of t^λ and define x_λ and y_λ as the following elements of RS_n

$$x_\lambda = \sum_{\sigma \in \text{Stab}_\lambda} \sigma \text{ and } y_\lambda = \sum_{\sigma \in \text{Stab}_\lambda} (-1)^{|\sigma|} \sigma$$

where $|\sigma|$ is the sign of σ . Recall that S_n acts on the right on tableaux by place permutations. For $t \in \text{Std}(\lambda)$, we define the associated element $d(t) \in S_n$ by

$$t^\lambda d(t) = t.$$

Then for pairs (s, t) of λ -tableaux, Murphy's standard basis and dual standard basis for RS_n consist of the elements

$$x_{st} = d(s)^{-1} x_\lambda d(t) \text{ and } y_{st} = d(s)^{-1} y_\lambda d(t).$$

3. PROJECTIVE MODULES

Recall that the blocks for $\mathbb{F}S_n$ are given by the Nakayama Conjecture (which is a Theorem). Murphy showed in [Mu83] how to describe the corresponding block idempotents in terms of the Jucys-Murphy idempotents E_t . Indeed, let $T = [t]$ be the class of $t \in \text{Std}(n)$ under \sim_p and consider for $\lambda \in \text{Par}(n)$ the following set \mathcal{T}_λ of tableaux

$$\mathcal{T}_\lambda := \{s \mid \text{for a } \sigma \in S_n \text{ there is a tableau of shape } \lambda \text{ in } [\sigma s]\}. \quad (4)$$

Let $[\lambda]$ be the class of λ under the equivalence relation on $\text{Par}(n)$ given by $\lambda \sim_p \mu$ if $\mathcal{T}_\lambda = \mathcal{T}_\mu$. Then Murphy showed in *loc. cit.* that $E_T := \sum_{t \in T} E_t$ and $E_{[\lambda]} := \sum_{t \in \mathcal{T}_\lambda} E_t$ lie in RS_n and that $\overline{E_{[\lambda]}} \in \mathbb{F}_p S_n$ is the block idempotent for the block given by $[\lambda]$. In particular the $\overline{E_{[\lambda]}}$'s are pairwise orthogonal and central in $\mathbb{F}_p S_n$ with sum 1.

Let $\mathbb{F}_p S_n\text{-mod}$ denote the category of finite dimensional $\mathbb{F}_p S_n$ -modules and let

$$\mathcal{I}_{n-1}^n : \mathbb{F}_p S_{n-1}\text{-mod} \rightarrow \mathbb{F}_p S_n\text{-mod}, M \mapsto \mathbb{F}_p S_n \otimes_{\mathbb{F}_p S_{n-1}} M$$

be the induction functor from $\mathbb{F}_p S_{n-1}\text{-mod}$ to $\mathbb{F}_p S_n\text{-mod}$.

Assume that $\lambda \in \text{Par}(n-1)$ is a subpartition of $\mu \in \text{Par}(n)$ and that $\mu \setminus \lambda$ consists of one node of residue i . Then Robinson's i -induction functor f_i is defined as

$$f_i : \mathbb{F}_p S_{n-1}\text{-mod} \rightarrow \mathbb{F}_p S_n\text{-mod}, M \mapsto \overline{E_{[\mu]}} \mathbb{F}_p S_n \otimes_{\mathbb{F}_p S_{n-1}} M.$$

Consider the following set $\mathcal{T}_{i,n}$ of tableaux classes of n -tableaux

$$\mathcal{T}_{i,n} := \{ [t] \mid s[n] = i \bmod p \text{ for some (any) } s \in [t] \}$$

and set $\overline{E_{i,n}} := \sum_{T \in \mathcal{T}_{i,n}} \overline{E_T}$. Then by Murphy's theory, $\overline{E_{i,n}}$ is an idempotent in $\mathbb{F}_p S_n$ and $\sum_i \overline{E_{i,n}} = 1$. We then have the following Lemma.

Lemma 2. *Suppose that M lies in the $[\lambda]$ -block of $\mathbb{F}_p S_{n-1}$. Then there is an isomorphism of $\mathbb{F}_p S_n$ -modules*

$$f_i M \cong \mathbb{F}_p S_n \overline{E_{i,n}} \otimes_{\mathbb{F}_p S_{n-1}} M.$$

Proof. Since the E_t 's sum to 1, we have that $E_{[\lambda]}$, viewed as an element of RS_n , is the sum of all E_s where s is obtained from a tableau in \mathcal{T}_λ by adding an addable node containing n . From this we deduce

$$\overline{E_{[\mu]}} = \overline{E_{[\mu]}} \overline{E_{[\lambda]}} = \overline{E_{i,n}} \overline{E_{[\lambda]}}.$$

On the other hand $\overline{E_{[\mu]}}$ is central in $\mathbb{F}_p S_n$ and so we get

$$f_i M \cong \mathbb{F}_p S_n \overline{E_{[\mu]}} \otimes_{\mathbb{F}_p S_{n-1}} M \cong \mathbb{F}_p S_n \overline{E_{i,n}} \overline{E_{[\lambda]}} \otimes_{\mathbb{F}_p S_{n-1}} M \cong \mathbb{F}_p S_n \overline{E_{i,n}} \otimes_{\mathbb{F}_p S_{n-1}} M$$

as claimed. \square

We next introduce the notation that allows us to generalize the Lemma to the 'divided powers'. This is an important topic of the paper. Assume therefore that μ is a p -restricted partition of n . Assume furthermore that all its ladders \mathcal{L}_k are of length $|\mathcal{L}_k|$ strictly less than p . The partition $\mu = (6, 5, 3, 1)$ considered above violates this condition, since for example $|\mathcal{L}_5| = 3$, whereas the partition $\nu = (4, 4, 3, 1)$ meets it. Its p -residue diagram is

$$\nu_{res} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & \\ \hline 1 & 2 & 0 & \\ \hline 0 & & & \\ \hline \end{array}$$

and the ladder lengths are 1, 1, 2, 2, 1, 2, all less than 3.

The following Lemma relates the condition on the ladder lengths to the condition of James' Conjecture on the modular reduction from the Hecke algebra $\mathcal{H}_n(\xi)$ of type A at a p 'th root of unity to $\mathbb{F}_p S_n$, mapping $\xi \mapsto 1$, see [Ja]. Recall that it states that the decomposition numbers for $\mathbb{F}_p S_n$ and the decomposition numbers for $\mathcal{H}_n(\xi)$ coincide when $n < p^2$.

Lemma 3. *Let $p > 2$. Assume that n is an integer for which James' conjecture for $\mathbb{F}_p S_n$ should hold, that is $n < p^2$. Then for all p -restricted partitions μ of n , all ladders have lengths strictly less than p .*

Proof. Suppose that $\mu \in \text{Par}_{res}(n)$. Let $\mathcal{L} = \mathcal{L}_k$ be a ladder for μ with top node (a_1, b_1) and bottom node (a_2, b_2) and suppose that it has length p , that

is $p = a_1 - a_2 + 1$. Then the subdiagram of $\mu_{lad, \leq k}$ consisting of the nodes (a, b) satisfying $a_1 \leq a \leq a_2$ and $b_2 \leq b \leq b_1$ is a core partition of shape

$$\lambda_{core} = (1 + (p - 1)^2, \dots, (p - 1) + 1, 1).$$

Its parts form an arithmetic series with constant term 1 and difference $p - 1$, and so its order is $\frac{p(p^2 - 2p + 3)}{2}$. On the other hand, we have that

$$\frac{p(p^2 - 2p + 3)}{2} \geq p^2. \quad (5)$$

Indeed, $\frac{p(p^2 - 2p + 3)}{2} < p^2$ is equivalent to $(p - 1)(p - 3) < 0$ which is impossible since we assume $p \neq 2$. The Lemma follows from inequality (5). \square

We shall from now on assume that $p > 2$ and that $\mu \in \text{Par}_{res}(n)$ has ladders \mathcal{L}_k , $k = 1, 2, \dots, m$ of order strictly less than p . Let the residue of the ladder \mathcal{L}_k be i_k .

Associated with μ we define positive integers n_0, \dots, n_m by $n_0 := 0$ and

$$n_k := |\mathcal{L}_1| + |\mathcal{L}_2| + \dots + |\mathcal{L}_k|$$

for $k = 1, \dots, m$. We introduce the ladder group $S_{\mathcal{L}_k} \leq S_n$ for μ as

$$S_{lad, \mu} := \prod_k S_{\mathcal{L}_k},$$

where $S_{\mathcal{L}_k}$ is the symmetric group on the letters $n_{k-1} + 1, \dots, n_k$.

We write $\mathcal{I}_{n_{k-1}}^{n_k}$ for the induction functor from the category of finite dimensional $\mathbb{F}_p S_{n_{k-1}}$ -modules to the category of finite dimensional $\mathbb{F}_p S_{n_k}$ -modules

$$\mathcal{I}_{n_{k-1}}^{n_k} : M \mapsto \mathbb{F}_p S_{n_k} \otimes_{\mathbb{F}_p S_{n_{k-1}}} M.$$

Generalizing $\mathcal{T}_{i,n}$ we introduce the set $\mathcal{T}^{\mathcal{L}_k}$ of tableaux classes for S_{n_k} as

$$\mathcal{T}^{\mathcal{L}_k} := \{ [T] \mid t[j] = i_k \pmod p \text{ for } t \in [T] \text{ and } j = n_{k-1} + 1, \dots, n_k \}.$$

This gives rise to the following idempotents

$$E^{\mathcal{L}_k} := \sum_{T \in \mathcal{T}^{\mathcal{L}_k}} E_T \in R S_{n_k}, \quad \overline{E^{\mathcal{L}_k}} \in \mathbb{F}_p S_{n_k}.$$

Now since we are assuming $|\mathcal{L}_k| < p$, we can define another idempotent

$$e_k := \frac{1}{|\mathcal{L}_k|!} \sum_{\sigma \in S_{\mathcal{L}_k}} \sigma \in \mathbb{F}_p S_{n_k}.$$

We combine it with $\overline{E^{\mathcal{L}_k}}$ to define

$$\overline{E^{(\mathcal{L}_k)}} := \overline{E^{\mathcal{L}_k}} e_k \in \mathbb{F}_p S_{n_k}, \quad \tilde{e}_\mu := \prod_k \overline{E^{(\mathcal{L}_k)}}.$$

Lemma 4. $\overline{E^{(\mathcal{L}_k)}}$ and \tilde{e}_μ are idempotents of $\mathbb{F}_p S_{n_k}$ and $\tilde{e}_\mu = E_{[\mu]_{lad}} \prod_k e_k$.

Proof. By Lemma 1 of [RH3] the two factors of $\overline{E^{(\mathcal{L}_k)}}$ commute and so it is indeed an idempotent. Moreover, we have that $\overline{E^{(\mathcal{L}_k)}}$ commutes with $\mathbb{F}_p S_{n_{k-1}}$ and so all factors of \tilde{e}_μ commute and it also an idempotent. The last claim also follows from this. \square

We now get our divided power induction functor as

$$f_i^{(|\mathcal{L}_k|)} : \mathbb{F}_p S_{n_{k-1}}\text{-mod} \rightarrow \mathbb{F}_p S_{n_k}\text{-mod} \quad (6)$$

$$M \mapsto \mathbb{F}_p S_{n_k} \overline{E(\mathcal{L}_k)} \otimes_{\mathbb{F}_p S_{n_{k-1}}} M. \quad (7)$$

By Lemma 2, we have that if $|\mathcal{L}_k| = 1$ then $f_i^{(|\mathcal{L}_k|)} = f_i$.

With this at hand, we can now formulate the definition of the $\mathbb{F}_p S_n$ -module $\widetilde{A(\mu)}$, mentioned in the introduction of the paper. It is defined as

$$\widetilde{A(\mu)} := f_{i_m}^{(|\mathcal{L}_m|)} \dots f_{i_2}^{(|\mathcal{L}_2|)} f_{i_1}^{(|\mathcal{L}_1|)} \mathbb{F}_p. \quad (8)$$

Theorem 1. *a). There is an isomorphism of $\mathbb{F}_p S_n$ -modules $\widetilde{A(\mu)} \cong \mathbb{F}_p S_n \tilde{e}_\mu$. In particular, $\widetilde{A(\mu)}$ is a projective $\mathbb{F}_p S_n$ -module.*

b) For certain integers $m_{\lambda\mu} \geq 0$ there is a triangular expansion of the form

$$\widetilde{A(\mu)} = P(\mu) \oplus \bigoplus_{\lambda, \lambda \triangleright \mu} P(\lambda)^{\oplus m_{\lambda\mu}}.$$

Proof. We first prove a). By definition $\widetilde{A(\mu)}$ is isomorphic to

$$\mathbb{F}_p S_{n_m} \overline{E(\mathcal{L}_m)} \otimes_{\mathbb{F}_p S_{n_{m-1}}} \dots \otimes_{\mathbb{F}_p S_{n_2}} \overline{E(\mathcal{L}_2)} \otimes_{\mathbb{F}_p S_{n_1}} \overline{E(\mathcal{L}_1)} \otimes_{\mathbb{F}_p S_{n_1}} \mathbb{F}_p.$$

Since $\overline{E(\mathcal{L}_k)}$ commutes with $\mathbb{F}_p S_{n_{k-1}}$ for all k , this simplifies to

$$\mathbb{F}_p S_n \prod_k \overline{E(\mathcal{L}_k)} = \mathbb{F}_p S_n \tilde{e}_\mu$$

as claimed.

We next show b). Assume that $P(\lambda)$ is a summand of $\widetilde{A(\mu)}$. Then we have that $\text{Hom}_{\mathbb{F}_p S_n}(\widetilde{A(\mu)}, D(\lambda)) \neq 0$ and hence $\text{Hom}_{\mathbb{F}_p S_n}(\widetilde{A(\mu)}, S(\lambda)) \neq 0$ since $D(\lambda)$ is a quotient of $S(\lambda)$ and $\widetilde{A(\mu)}$ is projective. On the other hand, by the definition of $\widetilde{A(\mu)}$ we have that

$$\text{Hom}_{\mathbb{F}_p S_n}(\widetilde{A(\mu)}, S(\lambda)) = \tilde{e}_\mu S(\lambda) = \prod_k e_k \overline{E_{[\mu]_{lad}}} S(\lambda). \quad (9)$$

We now show that $E_{[\mu]_{lad}} S(\lambda) \neq 0$ implies that $\lambda \triangleright \mu$. We view $E_{[\mu]_{lad}}$ as an element of $\mathbb{Q} S_n$ and get via Lemma 1, that in the expansion of it as a sum of E_t , only those t with $\text{Shape}(t) \triangleright \mu$ can appear. On the other hand, over \mathbb{Q} the standard basis $\{x_{s\lambda}, s \in \text{Std}(\lambda)\}$ for $S(\lambda)_\mathbb{Q} := S(\lambda) \otimes_R \mathbb{Q}$ may be replaced by the seminormal basis $\{\xi_{s\lambda}, s \in \text{Std}(\lambda)\}$, as defined in [Mu92] via $\xi_{s\lambda} = E_s x_{s\lambda}$, and since $E_t \xi_{s\lambda} \neq 0$ implies $\text{Shape}(t) = \lambda$ we indeed get the triangularity property of b).

To show that $P(\mu)$ occurs with multiplicity one in $\widetilde{A(\mu)}$, we set $\lambda = \mu$ in (9) and verify that $\tilde{e}_\mu S(\lambda)$ has dimension one over \mathbb{F}_p . Define first $M := E_{[\mu]_{lad}} S(\mu)$. According to Hu and Mathas's key observation in Lemma 4.1 of [HuMa], M is the μ_{lad} 'th generalized eigenspace for the action of L_k on $S(\mu)$. But $\sigma \in S_{lad, \mu}$ and $E_{[\mu]_{lad}}$ commute, as already mentioned above, and so in fact M is a $S_{lad, \mu}$ -module. Moreover, as an R -module it is free, being a submodule of the free R -module $S(\mu)$. Let us now determine its rank by

extending scalars from R to \mathbb{Q} . By Lemma 1 once again, we get that in the expansion of $E_{[\mu_{lad}]}$ as a sum of E_t 's, the occurring t with $Shape(t) = \mu$ are exactly those of the form $\sigma\mu_{lad}$ where $\sigma \in S_{lad,\mu}$ and hence, over \mathbb{Q} , we get a basis for M consisting of $\{\xi_{s\mu}\}$ where $s = \sigma\mu_{lad}$. In other words, M has dimension $|S_{lad,\mu}|$ over \mathbb{Q} and then also over R and \mathbb{F}_p .

Let us now check that the homomorphism φ given by

$$\mathbb{F}_p S_{lad,\mu} \rightarrow M, \quad \sigma \mapsto E_{[\mu_{lad}]} \sigma x_{\mu_{lad},\mu}$$

is injective. First of all, for $\sigma \in S_{lad,\mu}$ we know from Murphy's theory that

$$E_{[\mu_{lad}]} x_{\sigma\mu_{lad},\mu} = \sigma x_{\mu_{lad},\mu}$$

modulo higher terms, that is modulo $x_{s\mu}$ where $s \triangleright \sigma\mu_{lad}$. If $\sum_{\sigma \in S_{lad,\lambda}} \lambda_\sigma \sigma \neq 0$, we choose σ with $\sigma\mu_{lad}$ minimal subject to $\lambda_\sigma \neq 0$, and find from the above that the coefficient of $x_{\sigma\mu_{lad},\mu}$ in $\varphi(\sum_{\sigma \in S_{lad,\lambda}} \lambda_\sigma \sigma)$ is nonzero, proving that φ indeed is injective. By dimension comparison it is therefore bijective; in other words M is isomorphic to the regular module for $S_{lad,\mu}$.

From this we finally deduce that (9) has dimension one when $\lambda = \mu$, and b) is proved. \square

4. THE CONJECTURE AND THE LLT-ALGORITHM

Let us recall the Fock space \mathcal{F}_q associated with the representation theory of the Hecke algebra $\mathcal{H}_n(\xi)$ at a p 'th root of unity. As a $\mathbb{C}(q)$ -vector space, we have

$$\mathcal{F}_q := \bigoplus_{\lambda \in \text{Par}} \mathbb{C}(q)\lambda$$

where $\text{Par} := \bigcup_{n=0}^{\infty} \text{Par}_n$ with the convention that $\text{Par}_0 := \{\emptyset\}$. It is an integrable module for the quantum group $\mathcal{U}_q(\widehat{\mathfrak{sl}}_p)$, where we use the version of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_p)$ that appears for example in [LLT]. It is a $\mathbb{C}(q)$ -algebra on generators $e_i, f_i, i = 0, 1, \dots, p-1$ and k_h for h belonging to the Cartan subalgebra \mathfrak{h} of the associated Kac-Moody algebra, all subject to certain relations that we do not detail. Let us be more precise concerning the action $\mathcal{U}_q(\widehat{\mathfrak{sl}}_p)$ in \mathcal{F}_q . Suppose that $\mu \in \text{Par}(n)$. A node of μ is called removable if it can be removed from μ with the result being the diagram of a partition λ . Dually, that node is called an addable node of λ . It is called an i -node if its p -residue is i . Assume now that $\gamma = \mu \setminus \lambda$ is a removable i -node of μ . We then define

$$\begin{aligned} N_i^l(\gamma) &:= |\{\text{addable } i\text{-nodes to the left of } \gamma\}| - |\{\text{removable } i\text{-nodes to the left of } \gamma\}| \\ N_i^r(\gamma) &:= |\{\text{addable } i\text{-nodes to the right of } \gamma\}| - |\{\text{removable } i\text{-nodes to the right of } \gamma\}| \end{aligned}$$

The action of $e_i, f_i, i = 0, 1, \dots, p-1$ on \mathcal{F}_q is now given by the following formulas

$$f_i \lambda = \sum_{\mu \in \text{Par}(n), \gamma = \mu \setminus \lambda} q^{N_i^l(\gamma)} \mu, \quad e_i \mu = \sum_{\lambda \in \text{Par}(n-1), \gamma = \mu \setminus \lambda} q^{-N_i^r(\gamma)} \lambda \quad (10)$$

where γ runs over addable λ -nodes in the first sum, and over removable μ -nodes in the second sum. Note that since [LLT] use the duals of our Specht modules, the formulas for the action on \mathcal{F}_q that appear there are slightly different. There are similar formulas for the action of the other generators, but we leave them out.

Setting $\mathcal{M}_q := \mathcal{U}_q(\widehat{\mathfrak{sl}}_p) \emptyset$ we obtain an irreducible module for $\mathcal{U}_q(\widehat{\mathfrak{sl}}_p)$, the basic module, which is provided with a canonical basis/global crystal by Lusztig and Kashiwara's general theory. Indeed, let $u \mapsto \bar{u}$ be the usual bar involution of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_p)$, satisfying $\bar{q} = q^{-1}$, $\bar{q^h} = q^{-h}$, $\bar{f}_i = f_i$ and $\bar{e}_i = e_i$ for all relevant i . It induces an involution $m \mapsto \bar{m}$ of \mathcal{M}_q . For $k \in \mathbb{Z}$ we let $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$ be the usual Gaussian integer (with the convention $[0]_q = 0$), let $[k]_q! := [k]_q [k-1]_q \dots [1]_q$ and define the divided powers as $f_i^{(k)} := \frac{1}{[k]_q!} f_i^k$ and $e_i^{(k)} := \frac{1}{[k]_q!} e_i^k$. We may then define $\mathcal{U}_{\mathbb{Q}}^-$ as the $\mathbb{Q}[q, q^{-1}]$ -subalgebra of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_p)$ generated by $f_i^{(k)}, k = 1, 2, 3, \dots$ and introduce $\mathcal{M}_{\mathbb{Q}} := \mathcal{U}_{\mathbb{Q}}^- \emptyset$. Let $A := \{f(q)/g(q) \mid f(q), g(q) \in \mathbb{Q}[q], g(q) \neq 0\}$. Then A is a subring of $\mathbb{Q}(q)$ and we define L as the A -sublattice of \mathcal{F} generated by all $\lambda \in \text{Par}$. The following Theorem follows from Kashiwara and Lusztig's general theory.

Theorem 2. *There is a unique $\mathbb{Q}[q, q^{-1}]$ -basis $\{G(\lambda) \mid \lambda \in \bigcup_n \text{Par}_{\text{res}}(n)\}$ for $\mathcal{M}_{\mathbb{Q}}$, called the lower global crystal basis, satisfying*

$$a) \ G(\lambda) \equiv \lambda \text{ mod } qL, \quad b) \ \overline{G(\lambda)} = G(\lambda).$$

Recall now that Lascoux, Leclerc and Thibon introduced in [LLT] for $\mu \in \text{Par}_{\text{res}}(n)$ an element $A(\mu)$ of \mathcal{M}_q called “the first approximation to $G(\lambda)$ ”. It is defined as

$$A(\mu) := f_{i_m}^{(|\mathcal{L}_m|)} \dots f_{i_2}^{(|\mathcal{L}_2|)} f_{i_1}^{(|\mathcal{L}_1|)} \emptyset. \quad (11)$$

where $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m$ still are the ladders for μ with residues i_1, \dots, i_m . Based on this, they explain a recursive algorithm, the LLT-algorithm, that determines $G(\mu)$ in terms of $A(\lambda)$ where $\lambda \triangleright \mu$. The following is an immediate consequence of that algorithm.

Theorem 3. *There is an expansion of the form*

$$A(\mu) = G(\mu) + \sum_{\lambda, \lambda \triangleright \mu} n_{\lambda\mu} G(\lambda)$$

for certain $n_{\lambda\mu} \in \mathbb{Z}[q, q^{-1}]$ satisfying $\overline{n_{\lambda\mu}} = n_{\lambda\mu}$.

The main purpose of our paper is to study the following conjecture.

Conjecture 1. *Suppose that $n < p^2$. Then we have that*

$$n_{\lambda\mu}(1) = m_{\lambda\mu}$$

where $n_{\lambda\mu}$ is as in Theorem 3 and $m_{\lambda\mu}$ as in Theorem 1. In particular $n_{\lambda\mu}(1)$ is a nonnegative integer.

Remark. In the next section we give strong experimental evidence in favour of the Conjecture, see Theorem 8 below. We note at this point that in the cases that are covered by Theorem 8, we always have that $n_{\lambda\mu}$ is constant, that is $n_{\lambda\mu} = n_{\lambda\mu}(1)$. But experimental evidence beyond the cases that are covered by Theorem 8 suggest that, as predicted by the Conjecture, $n_{\lambda\mu}(1)$ is nonnegative even when $n_{\lambda\mu}$ is nonconstant, for instance for $p = 5$ we have checked that $n_{\lambda\mu}(1) \geq 0$ for all $n < 5^2$. Our motivation for imposing the condition $n < p^2$ in the Conjecture comes from this experimental evidence.

Let $\mathcal{G}(n)$ be the Grothendieck group of finitely generated $\mathbb{F}_p S_n$ -modules, and let $\mathcal{K}(n)$ be the Grothendieck group of finitely generated projective $\mathbb{F}_p S_n$ -modules. If M is a (projective) $\mathbb{F}_p S_n$ -module, we denote by $[M]$ its image in $\mathcal{G}(n)$ ($\mathcal{K}(n)$). We have that $\mathcal{G}(n)$ and $\mathcal{K}(n)$ are free Abelian groups with bases given by $\{[D(\mu)]\}$ and $\{[P(\mu)]\}$ for $\mu \in \text{Par}_{res}(n)$. There is a non-degenerate bilinear pairing (\cdot, \cdot) between $\mathcal{G}(n)$ and $\mathcal{K}(n)$ which is given by $([P], [M]) = \dim \text{Hom}_{\mathbb{F}_p S_n}(P, M)$. Using it, we have the following formula for the decomposition number for $\mathbb{F}_p S_n$

$$d_{\lambda\mu} = ([P(\mu)], [S(\lambda)]).$$

These constructions and definitions can also be carried out for the Hecke algebra $\mathcal{H}_n(\xi)$, and we shall in general use a superscript 'Hecke' for the corresponding quantities.

Our interest in Conjecture 1 comes from the following Theorem.

Theorem 4. *Suppose that Conjecture 1 is true. Then James' Conjecture holds.*

Proof. For $\mu \in \text{Par}_{res}(n)$ and $\lambda \in \text{Par}(n)$ we define $A(\mu)_\lambda \in \mathbb{Q}[q, q^{-1}]$ as the coefficient of λ in the expansion of $A(\mu)$. We first check that

$$(A(\mu)_\lambda)(1) = \dim_{\mathbb{F}_p}(\tilde{e}_\mu S(\lambda)). \quad (12)$$

To calculate $(A(\mu)_\lambda)(1)$ we put $q = 1$ in the formula (10) to arrive at $(f_{i_m}^{|\mathcal{L}_m|} \dots f_{i_2}^{|\mathcal{L}_2|} f_{i_1}^{|\mathcal{L}_1|} \emptyset)_\lambda(1)$. But this is exactly the number of tableaux in $[\mu_{lad}]$ of shape λ , since we may identify $\text{Std}(\lambda)$ with sequences of partitions, corresponding to the action of the f_i 's at $q = 1$. On the other hand we have

$$E_{[\mu_{lad}]} S(\lambda) = \sum_{t \in [\mu_{lad}]} E_t S(\lambda) = \sum_{t \in [\mu_{lad}], \text{Shape}(t) = \lambda} E_t S(\lambda).$$

In [Mu83] a basis is given for this space, from which we get that its dimension is the number of tableaux in $[\mu_{lad}]$ of shape λ , see page 263 of *loc. cit.* Now clearly $S_{lad, \mu}$ acts faithfully on the tableaux in $[\mu_{lad}]$ of shape λ but it also acts faithfully on $E_{[\mu_{lad}]} S(\lambda)$, as can be seen by arguing as in Theorem 1 b) and so (12) follows by symmetrizing over $S_{lad, \mu}$.

Let us now assume that Conjecture 1 holds and let $(a_{\lambda\mu}) := (n_{\lambda\mu}(1))^{-1}$. Then we have that

$$G(\mu)(1) = A(\mu)(1) + \sum_{\lambda, \lambda \triangleright \mu} a_{\lambda\mu} A(\lambda)(1), \quad [P(\mu)] = [\widetilde{A(\mu)}] + \sum_{\lambda, \lambda \triangleright \mu} a_{\lambda\mu} [\widetilde{A(\lambda)}] \quad (13)$$

where the last equality takes place in $\mathcal{K}(n)$. We get from it that

$$d_{\tau\mu} = ([\widetilde{A(\mu)}], [S(\tau)]) + \sum_{\lambda, \lambda \triangleright \mu} a_{\lambda\mu} ([\widetilde{A(\lambda)}], [S(\tau)])$$

which, using equation (12) and the definition of (\cdot, \cdot) , can be rewritten as

$$d_{\tau\mu} = (A(\mu)_\tau)(1) + \sum_{\lambda, \lambda \triangleright \mu} a_{\lambda\mu} (A(\lambda)_\tau)(1) = (G(\mu)_\tau)(1)$$

where we for the last equality used the first equality of (13). Finally, by Ariki's proof of the main Conjecture of [LLT], we know that $(G(\mu)_\tau)(1) = d_{\tau\mu}^{hecke}$. The Theorem is proved. \square

5. PARTIAL VERIFICATION OF CONJECTURE 1

We still set $(a_{\lambda\mu}) = (n_{\lambda\mu}(1))^{-1}$ and consider for $\mu \in \text{Par}_{res}(n)$ the element $\mathcal{P}(\mu) \in \mathcal{K}(n)$ given by

$$\mathcal{P}(\mu) := [\widetilde{A(\mu)}] + \sum_{\lambda, \lambda \triangleright \mu} a_{\lambda\mu} [\widetilde{A(\lambda)}]. \quad (14)$$

Assume from now on that $n < p^2$. In order to prove Conjecture 1 we must show for all $\lambda, \mu \in \text{Par}_{res}(n)$ that $(\mathcal{P}(\mu), [D(\tau)]) = \delta_{\mu\tau}$ or equivalently

$$\dim_{\mathbb{F}_p}(\widetilde{e}_\mu D(\tau)) + \sum_{\lambda, \lambda \triangleright \mu} a_{\lambda\mu} \dim_{\mathbb{F}_p}(\widetilde{e}_\lambda D(\tau)) = \delta_{\mu\tau} \quad (15)$$

since we would then have that $\mathcal{P}(\mu) = [P(\mu)]$. The number of terms in the summation of (15) is relatively small, so in order to verify these equations, we essentially need a way of determining the dimension of the symmetrized generalized weight spaces $\widetilde{e}_\lambda D(\tau)$, or equivalently, the rank of $\langle \cdot, \cdot \rangle_\tau$ on the restriction to $\widetilde{e}_\lambda S(\tau)$.

Let \mathcal{R}_n be the cyclotomic KLR-algebra (Khovanov-Lauda-Rouquier) of type A . It is an \mathbb{F}_p -algebra on generators

$$\{e(\mathbf{i}) \mid \mathbf{i} \in (\mathbb{F}_p)^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$$

subject to a list of relations that we do not detail here. But in their seminal paper [BK], Brundan and Kleshchev proved the existence of an algebra isomorphism $\mathbb{F}_p S_n \cong \mathcal{R}_n$ that maps the generators of \mathcal{R}_n to concretely constructed elements of $\mathbb{F}_p S_n$, that are denoted the same way.

Let us now focus on the elements $\psi_1, \dots, \psi_{n-1} \in \mathbb{F}_p S_n$ that are constructed in [BK] via an adjustment of the 'intertwining elements' $\phi_1, \dots, \phi_{n-1} \in \mathbb{F}_p S_n$. In [RH3] we found a realization of these intertwining elements within the theory of Young's seminormal form. Indeed, $\phi_i = \sigma_i - \frac{1}{h_L}$ where $\frac{1}{h_L} = \frac{1}{L_{i-1} - L_i}$ is defined in Lemma 5 and the intertwining property is given in Lemma 7 of *loc. cit.*

Define $S(\lambda)_{\mathbb{Q}} := S(\lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\{\xi_{st} \mid (s, t) \in \text{Std}(\lambda)^2, \lambda \in \text{Par}(n)\}$ be the seminormal basis for $\mathbb{Q} S_n$, as introduced in [Mu92]. Let $\{x_{st} \mid (s, t) \in \text{Std}(\lambda)^2, \lambda \in \text{Par}(n)\}$ be as above and set $(\mathbb{Z} S_n)^{>\lambda} := \text{span}_{\mathbb{Z}}\{x_{st} \mid \text{Shape}(s) > \lambda\}$. Then $(\mathbb{Z} S_n)^{>\lambda}$ is an ideal of $\mathbb{Z} S_n$ and $\{x_{s\lambda} + (\mathbb{Z} S_n)^{>\lambda} \mid s \in \text{Std}(\lambda)\}$ is the standard basis for $S(\lambda)$. On the other hand, $\{\xi_{s\lambda} \mid s \in \text{Std}(\lambda)\}$ is a basis for $S(\lambda)_{\mathbb{Q}}$. The action of S_n on the standard basis $\{x_{s\lambda}\}$ is given by a recursion using the Garnir relations, whereas the action of S_n on $\{\xi_{s\lambda}\}$ is given by the following formulas, that appear for example in Theorem 6.4 of [Mu93] for general q (but note the sign error there: the expression for h should be replaced by $-h$).

Theorem 5. *Let $h = c_s(i-1) - c_s(i)$ be the radial distance between the $i-1$ and i -nodes of $s \in \text{Std}(\lambda)$. Let $t := s\sigma_i$ where still $\sigma_i = (i-1, i)$. Then the*

action of σ_i on $\xi_{s\lambda}$ is given by the formulas

$$\sigma_i \xi_{s\lambda} := \begin{cases} \xi_{s\lambda} & \text{if } h = -1 \text{ (} i-1 \text{ and } i \text{ are in same row)} \\ -\xi_{s\lambda} & \text{if } h = 1 \text{ (} i-1 \text{ and } i \text{ are in same column)} \\ -\frac{1}{h} \xi_{s\lambda} + \xi_{t\lambda} & \text{if } h > 1 \text{ (} i-1 \text{ is above } i) \\ -\frac{1}{h} \xi_{s\lambda} + \frac{h^2-1}{h^2} \xi_{t\lambda} & \text{if } h < -1 \text{ (} i-1 \text{ is below } i). \end{cases} \quad (16)$$

On the other hand, these formulas take place in $S(\lambda)_{\mathbb{Q}}$ and therefore do not immediately help us in the modular setting. But note that by [Mu92] we have that

$$\xi_{\lambda\lambda} = x_{\lambda\lambda} \pmod{(\mathbb{Q}S_n)^{>\lambda}}$$

and so, using the formulas on a reduced expression $d(s) = \sigma_{i_1} \dots \sigma_{i_N}$, we can express the standard basis element $x_{s\lambda}$ as a linear combination of the seminormal basis elements $\xi_{s\lambda}$, with coefficients in \mathbb{Q} .

Theorem 6. *Let T be a tableau class and suppose that $x \in E_T S(\lambda)$ (defined over R). Then x can be written in the above way as $x = \sum_{t \in T} a_t \xi_{t\lambda}$. Moreover, the action of the intertwiner ϕ_i is given by $\phi_i x = \sum_{t \in T} a_t \phi_i^m \xi_{t\lambda}$ where*

$$\phi_i^m \xi_{s\lambda} := \begin{cases} 0 & \text{if } |h| = 1 \\ \xi_{t\lambda} & \text{if } h > 1 \text{ and } s \approx_p t \\ \frac{h^2-1}{h^2} \xi_{t\lambda} & \text{if } h < 1 \text{ and } s \approx_p t \\ (-1 - \frac{1}{h}) \xi_{s\lambda} + \xi_{t\lambda} & \text{if } h > 1 \text{ and } s \sim_p t \\ (-1 - \frac{1}{h}) \xi_{s\lambda} + \frac{h^2-1}{h^2} \xi_{t\lambda} & \text{if } h < -1 \text{ and } s \sim_p t \end{cases} \quad (17)$$

for $t := s\sigma_i$. We say that the first three cases of these formulas are the ‘regular’ ones whereas the last two cases are the ‘singular’ ones.

Proof. The first statement is a consequence of the realization of the tableau class idempotent E_T as $E_T = \sum_{t \in T} E_t$ and the fact that $E_t S(\lambda)_{\mathbb{Q}} = \mathbb{Q} \xi_{t\lambda}$.

In order to prove the second statement, we need to recall the construction of ϕ_i from [RH3]. Let $S := [s], T := [t]$ and suppose first that $S \neq T$. Choose arbitrarily $t \in T$ and define $c_T(i-1) := c_t(i-1) \in R$ and $c_T(i) := c_t(i) \in R$ and set $h_T(i) := c_T(i-1) - c_T(i)$. Although $h_T(i) \in R$ depends on the choice of $t \in T$, we showed in [RH3] that for any $a \in E_T S(\lambda)$, we have that $(L_{i-1} - L_i - h)^N a$ belongs to $pE_T S(\lambda)$ for N sufficiently big, independently of the choice of t . Then $\frac{1}{L_{i-1} - L_i}$ is defined on $\overline{E_T S(\lambda)}$ via the corresponding geometric series. To be precise, for $a \in \overline{E_T S(\lambda)}$ we set

$$\frac{1}{L_{i-1} - L_i} a := \frac{1}{h_T(i)} \sum_k (-1)^k \left(\frac{L_{i-1} - L_i - h_T(i)}{h_T(i)} \right)^k a \quad (18)$$

Finally, ϕ_i is defined as $\phi_i := \sigma_i - \frac{1}{L_{i-1} - L_i}$.

Actually we showed in [RH3] that $(L_{i-1} - L_i - h_T(i))^N a \in pS(\lambda)$ for large N by expanding $a \in E_T S(\lambda)$ in the seminormal basis $\xi_{t\lambda}$ where indeed

$$(L_{i-1} - L_i - h_T(i))^N \xi_{t\lambda} = (c_t(i-1) - c_t(i) - h_T(i))^N \xi_{t\lambda} \in pS(\lambda)$$

for $N \gg 0$. In other words, the series (18) is calculated by lifting $a \in \overline{E_T S(\lambda)}$ to $a \in E_T S(\lambda)$, expanding a in $\xi_{s\lambda}$, applying the series to each term

and finally reducing modulo p . But

$$\frac{1}{h_T(i)} \sum_k (-1)^k \left(\frac{c_{i-1}(t) - c_i(t) - h_T(i)}{h_T(i)} \right)^k \xi_{s\lambda} \quad (19)$$

which is equal to $\frac{1}{c_{i-1}(t) - c_i(t)} \xi_{s\lambda}$ and from the regular cases of (17) follow from Theorem 16.

The singular cases are easier to handle, since we then have $\phi_i = \sigma_i - 1$ and so we may finish the proof of the Theorem by applying Theorem 16 once again. \square

Theorem 7. *Let $\lambda \in \text{Par}(n)$ and let $\{\xi_{s\lambda} \mid s \in \text{Std}(\lambda)\}$ be the seminormal basis for $S(\lambda)_{\mathbb{Q}}$. Then we have that*

$$\langle \xi_{s\lambda}, \xi_{s\lambda} \rangle_{\lambda} = \gamma_s.$$

Proof. This is contained in Murphy's papers where it is shown by induction. The induction basis is given by $\xi_{\lambda\lambda} = x_{\lambda\lambda}$ and the induction step by Young's seminormal form (16). \square

With the above Theorems at our disposal we can now describe an algorithm for calculating $\dim_{\mathbb{F}_p} \tilde{e}_{\lambda} D(\tau)$, or equivalently the rank of $\langle \cdot, \cdot \rangle_{\tau}$ on $\tilde{e}_{\lambda} S(\tau)$.

Algorithm.

Step 1. Determine the set $S^1 := \{s \in [\lambda_{lad}] \mid \text{Shape}(s) = \tau\}$. As indicated above, S^1 can be read off from the calculation of the first approximation of $A(\mu)$ at $q = 1$, since the successive actions of f_i may be viewed as producing tableaux rather than partitions.

Step 2. Write the elements of $S^2 := \{d(s) \mid s \in S^1\} \subset S_n$ as reduced products of simple transpositions σ_i . The longest element of S_n has length $l(w_0) = n(n-1)/2$, and so each of the reduced products has less than $n(n-1)/2$ terms.

Step 3. For each $d(s) = \sigma_{i_1} \dots \sigma_{i_k}$ from step 3, calculate $\phi_{i_k} \dots \phi_{i_1} \xi_{\lambda\lambda}$ using (17). By Theorem 2 and Lemma 9 of [RH3], we get in this way a basis for $e_{[\lambda_{lad}]} S(\tau)$. The basis elements are given as linear combinations of seminormal basis elements. The number of terms $\xi_{t\lambda}$ in this expansion will be less than 2^B where B is the number of indices in the reduced expression for $d(s)$ that involve the singular cases of (17).

Step 4. Symmetrize each basis element from the previous step with respect to the ladder group $S_{lad,\lambda}$, to get a basis for $\tilde{e}_{\lambda} S(\tau)$.

Step 5. Calculate the matrix of the form $\langle \cdot, \cdot \rangle_{\tau}$ on $\tilde{e}_{\lambda} S(\tau)$ with respect to the basis given in the previous step. Since the basis elements are expanded in terms of the seminormal basis, this step now follows easily from the previous Theorem. The matrix will have values in R although the coefficients of the expansions are rational.

Step 6. Reduce the matrix modulo p and determine its rank.

Remark. The algorithm can also be implemented using the classical Young's seminormal form, that is the formulas (16). On the other hand, that algorithm will be much less efficient with expansions that grow too fast. In fact, the main point of our algorithm, as presented above, is that the indices of

the reduced expressions will mostly correspond to the regular cases of (17), thus reducing, as much as possible, the doubling up of terms.

Example. Suppose that $p = 3$. We verify Conjecture 1 for S_5 using our algorithm. Although the LLT-algorithm does not involve any subtractions in this example, the example is still big enough to illustrate our algorithm.

We have that

$$\text{Par}_{res}(5) = \{\{[3, 2], [3, 1^2], [2^2, 1], [2, 1^3], [1^5]\}\}.$$

The ladder groups are

$$S_{lad,[3,2]} = S_{lad,[3,1^2]} = \{(3, 4)\}, \quad S_{lad,[2^2,1]} = S_{lad,[2,1^3]} = S_{lad,[1^5]} = 1.$$

The first approximations are

$$\begin{aligned} A([3, 2]) &= [3, 2] + q[4, 1], & A([3, 1^2]) &= [3, 1^2], \\ A([2^2, 1]) &= [2^2, 1] + q[5], & A([2, 1^3]) &= [2, 1^3] + q[2^2, 1], \\ A([1^5]) &= [1^5] + q[3, 2]. \end{aligned}$$

From this we conclude, as already mentioned above, that $G(\lambda) = A(\lambda)$ for all $\lambda \in \text{Par}_{res}(5)$. Thus, Conjecture 1 is in this case the affirmation that $P(\lambda) = \widetilde{A}(\lambda)$ for all $\lambda \in \text{Par}_{res}(5)$, or by the above that

$$\dim \tilde{e}_\lambda D(\tau) = \delta_{\lambda\tau} \quad \text{for all } \lambda, \tau \in \text{Par}_{res}(5).$$

We calculate the rank of $\langle \cdot, \cdot \rangle_\tau$ on each symmetrized weight space $\tilde{e}_\lambda S(\tau)$. Recall that $\dim \tilde{e}_\lambda S(\tau)$ can be read off from the first approximation $A(\lambda)$. For example we have that $\dim \tilde{e}_{[3,2]} S([4, 1]) = 1$ since the coefficient q evaluates to 1, although the eigenspace $\tilde{e}_{[3,2]} S([4, 1])$ is irrelevant to us since $[4, 1] \notin \text{Par}_{res}(5)$.

By going through the first approximations, we see that the relevant eigenspaces are $\tilde{e}_{[2,1^3]} S([2^2, 1])$ and $\tilde{e}_{[1^5]} S([3, 2])$, both of dimension one. We verify that the ranks of the corresponding forms are zero, or equivalently that the forms are zero.

We first consider $\tilde{e}_{[2,1^3]} S([2^2, 1])$. The residue diagram $res_{[2,1^3]}$ of $\lambda := [2, 1^3]$ is

$$res_{[2,1^3]} = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & \\ \hline 1 & \\ \hline 0 & \\ \hline \end{array} \quad t := \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array} \quad res_\tau = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 & \\ \hline \end{array}$$

and so we have $i_{lad,[2,1^3]} = (0, 1, 2, 1, 0)$. The only tableau of shape $\tau := [2^2, 1]$ in the ladder class of λ is therefore t as given above. We have $d(t) = (4, 5) = \sigma_5$ and so we get from formula (17) that the basis for $\tilde{e}_{[2,1^3]} S([2^2, 1])$ is $\{\phi_5 \xi_{\lambda\lambda}\} = \{\xi_{t\lambda}\}$. Finally, by Theorem 7 we get that $\langle \xi_{t\lambda}, \xi_{t\lambda} \rangle_\lambda = 3 = 0 \pmod 3$, as claimed.

We next consider $\tilde{e}_{[1^5]} S([3, 2])$ where we basically proceed as before. The residue diagram $res_{[1^5]}$ is

$$res_{[1^5]} := \begin{array}{|c|} \hline 0 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 2 \\ \hline \end{array} \quad s := \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array} \quad res_s := \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & \\ \hline \end{array}$$

and we have $\underline{i}_{lad,[2,1^3]} = (0, 2, 1, 0, 2)$. The only tableau of shape $\nu := [3, 2]$ in the ladder class of $[1^5]$ is s as given above. We have $d(s) = (3, 4)(4, 5)(2, 3) = \sigma_4\sigma_5\sigma_3$ and so we get from formula (17) that the basis for $\tilde{e}_{[1^5]}S([3, 2])$ is

$$\{\phi_3\phi_5\phi_4\xi_{\lambda\lambda}\} = \{\xi_{s\lambda}\}$$

and then by Theorem 7 we get $\langle \xi_{s\lambda}, \xi_{s\lambda} \rangle_\lambda = 3 = 0 \pmod 3$, as claimed. This concludes the verification of Conjecture 1, and then by Theorem 4 also of James's Conjecture, in this case.

We have implemented the algorithm using the GAP-system and have found the following results, that without doubt can be improved on.

Theorem 8. *If $p = 3$ then Conjecture 1 is true.*

If $p = 5$ then Conjecture 1 is true for $n < 16$.

If $p = 7$ then Conjecture 1 is true for $n < 19$.

If $p = 11$ then Conjecture 1 is true for $n < 22$.

If $p = 13$ then Conjecture 1 is true for $n < 22$.

Remark. The Theorem should, via Theorem 4, provide decomposition numbers for $\mathbb{F}_p S_n$ that were previously unknown.

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